CONCERNING GREEN'S THEOREM AND THE CAUCHY–RIEMANN DIFFERENTIAL EQUATIONS

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Consider a region R, bounded by a continuous closed curve C, and lying between the lines $y = a$ and $y = b$. If the curve C is such that lines parallel to the x-axis cut it in only a finite number of points we shall show that:

If $f(xy)$ and $f'_x(xy)$ be defined for all those points within R lying on the lines $y = y_k$, where the points y_k are everywhere dense in (ab), and if, $f(xy)$ being defined for all points of C, the integrals $\int_C f(xy)\,dy$ and $\int_R\int f'_x(xy)\,dx\,dy$ exist in Riemann's sense, then*

$$\int_C f(xy)\,dy = \int_R\int f'_x(xy)\,dx\,dy. \tag{1}$$

It will suffice to give the proof in the case $y = y_k$ cuts C in at most two points $x_i\ x_\iota$.

We have by definition,

$$\int_C f(xy)\,dy = \lim \Sigma\Big[f(x_i y_k) - f(x_\iota y_k)\Big]\Delta\, y_k, \qquad x_i \geqq x_\iota.$$

But

$$f(x_i\, y_k) - f(x_\iota\, y_k) = f(x_i\, y_k) - f(x_{i+1}\, y_k) + f(x_{i+2}\, y_k) - f(x_{i+\iota}\, y_k) + \cdots + f(x_{\iota-1}\, y_k) - f(x_\iota\, y_k)$$

identically, which on applying the Mean Value Theorem becomes:

$$f'_x(x'_i\, y_k)\Delta x_i + f'_x(x'_{i+1}\, y_k)\Delta x_{i+1} + \cdots + f'_x(x'_{\iota-1}\, y_k)\Delta x_\iota,$$

where $\qquad x_s < x'_s < x_{s+1} \quad \text{and} \quad \Delta x_s = x_{s+1} - x_s,$

so that

$$\int_C f(xy)\,dy = \lim \Sigma_R \Sigma\, f'_x(x'_s y_k)\,\Delta x_s\,\Delta y_k. \tag{2}$$

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*For a curve of type C, or more generally for any curve whose ordinate $y = \phi(t)$ is a continuous function of *limited variation* of t, a sufficient condition for the existence of this integral is that $f(xy)$ be continuous on C. *Cf.* Vallée-Poussin: *Cours d'Analyse*, I, p. 313.

The right-hand side is, by hypothesis, equal to $\int_R\int f'_x(xy)\,dx\,dy$, which proves equation (1).

In case $y = y_k$ cuts C in $2m$ points we should arrive at (2) by applying the identity and Mean Value Theorem to each of the m segments of $y = y_k$ lying inside of C.

We have hitherto supposed that f'_x was limited in R. If we suppose however that $|f'_x(xy)| < K$ on the lines $y = y_k$ but becomes infinite for point sets lying on lines $y = \bar{y}_k$ of the set *complementary* to $[y_k]$ in (ab), then (1) will still hold if $f'_x(xy)$ be *improperly* integrable over R.

The proof is simple: Since the point set over which f'_x becomes infinite must be of content zero, if we write (2) in the form

$$\int_C f(xy)dy = \lim\ \Sigma\Sigma'\, f'_x(x'_s\, y_k)\,\Delta x_s\ \Delta y_k\ +\ \lim\ \Sigma\Sigma''\, f'_x(x'_s y_k)\,\Delta x_s\,\Delta y_k,$$

where the second double sum applies to those rectangles in which f'_x becomes infinite and the first applies to the remaining rectangles, we see that the second limit will be zero since $|f'_x(x'_s\, y_k)| < K$ while the first will be the improper integral $\int_R\int f'_x dx\,dy$.

In conclusion we note that if the theory of analytic functions be founded on the Cauchy-Riemann differential equations

$$\left.\begin{aligned} u'_x &= v'_y \\ u'_y &= -\,v'_x \end{aligned}\right\}, \tag{3}$$

to establish the fundamental relation

$$\int_C (u + iv)(dx + idy) = 0 \qquad\qquad \Big[u + iv = f(z)\Big],$$

it will suffice to postulate that u and v are single valued and continuous and that the derivatives in (3) are limited, integrable, and defined over sets of parallel lines $x = [x_k]$, $y = [y_k]$ which are everywhere dense in the region. The existence of $f'(z)$ need not be postulated for any point of the region.

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